Problems around even-hole-free graphs

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based on joint work with Nicolas Trotignon

November 17, 2020

The world of *hereditary* graph classes

A piece of history...

- Optimization problems are NP-hard in general, but become polynomially solvable when some configurations are excluded
- A wonderful Robertson and Seymour's **graph minor theory** : for graphs closed under vertex/edge deletion and edge contraction
- How about graphs closed **only under vertex deletion**?

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Many interesting classes of graphs can be characterized as being $\mathcal{F}$-free

The world of hereditary graph classes

CHORDAL GRAPHS

- G if chordal if G contains **no hole**
  - hole : induced cycle of length ≥ 4
- everything is easy

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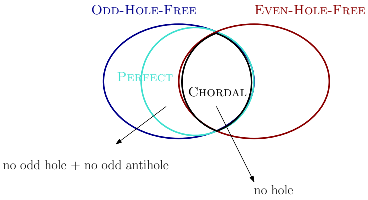
Perfect graphs vs even-hole-free graphs



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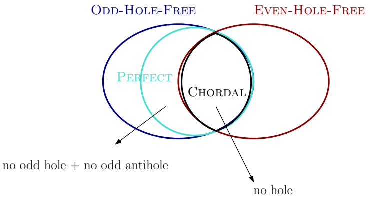

EVEN-HOLE FREE GRAPHS

- appear in the study of Strong Perfect Graph Conjecture
- it is structurally similar to perfect graphs
- even-hole-free = no even hole + no antihole of length ≥ 6
- decomposition theorem and recognition algorithm are known

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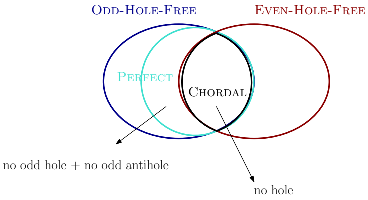

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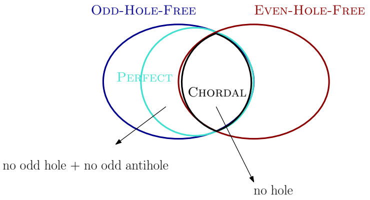

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Remark. For more about them, a survey by Kristina Vušković.

Even-hole-free graphs: go to a smaller world

- What to do?
  - What to study?
  - What to exclude?

What to study?

Bounding parameters? for ex. tree-width, rank-width, ...

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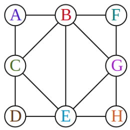
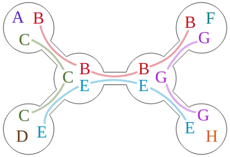
Tree decomposition & Tree-width

- Tree-width : a parameter measuring how far is a graph G from a tree





figures taken from https://commons.wikimedia.org/wiki/User:David_Eppstein/Gallery

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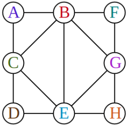

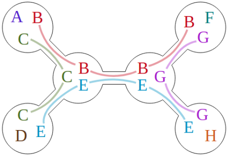

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  - The *width* $\mathcal{T}$ is the size of the largest bag minus 1
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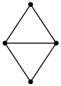
- small tree-width is good $\rightarrow$ many graph problems are easy

What to exclude?

- Excluding diamond $\rightarrow$ (even hole, diamond)-free graphs



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Theorem ([Adler, et al., 2017])

There exists a family of (even hole, triangle)-free graphs with arbitrarily large rank-width *(a graph parameter similar to tree-width)*

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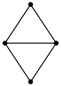

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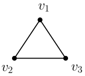

triangle

Theorem ([Cameron, et al., 2018])

Every (even hole, triangle)-free graph has tree-width at most 5

A question...

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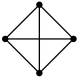
- But... how about excluding K_4?



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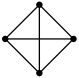


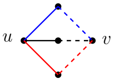
- But... how about excluding K_4?
- No ☹

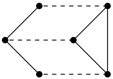
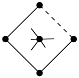
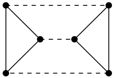
PART 1

Layered wheel

Truemper configurations

The following graphs are called Truemper configurations

- Theta : 

- Pyramid : 
- Wheel : 
- Prism : 

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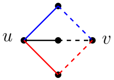
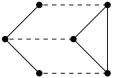
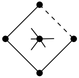
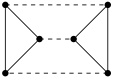
- Theta : 

- Pyramid : 
- Wheel : 
- Prism : 

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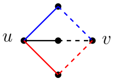
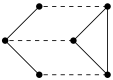
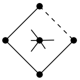
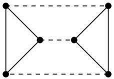
- Theta : 

- Pyramid : 
- Wheel : 
- Prism : 

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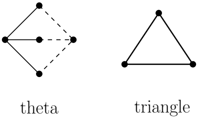
- Many studies about excluding Truemper configuration (see a survey by Kristina Vušković for more about them).

The class of (Theta, triangle)-free graphs



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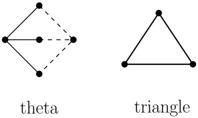
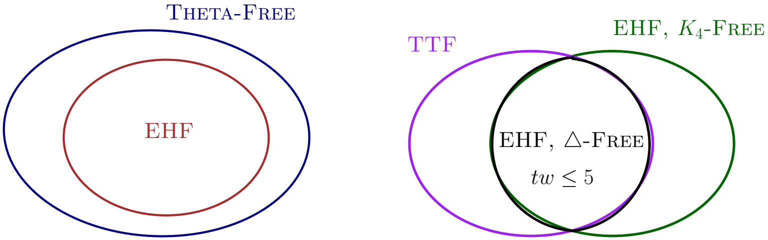
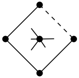


This class is "close" to the class of even-hole-free graphs



TTF graphs vs EHF graphs with no K_4

There are similarities in the structure of wheels



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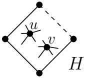

2-wheel

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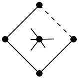
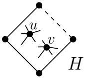


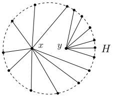
wheel



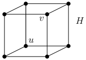
2-wheel

Structure of 2-wheels with **non-adjacent** centers:

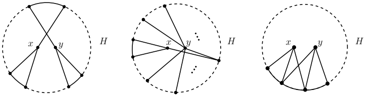
- In (even hole, triangle)-free : always nested
- In (theta, triangle)-free : nested, except the cube
- In (even hole, K_4)-free : nested, with several exceptions



nested wheel



cube



several exceptions

TTF layered wheel $G_{\ell,k}$, $\ell \geq 1, k \geq 4$

(Theta, Triangle)-Free Layered Wheel

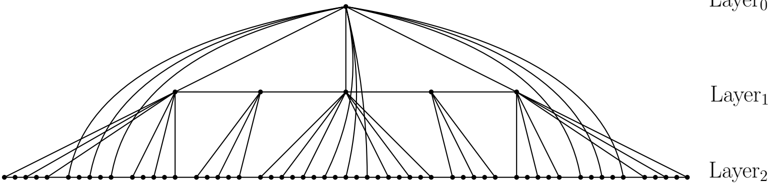
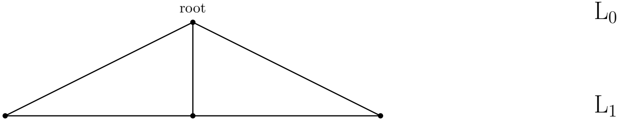


Figure: TTF layered wheel $G_{2,4}$

TTF layered wheel construction

$G(\ell, k)$, with $\ell = 2$ and $k = 4$

TTF layered wheel construction



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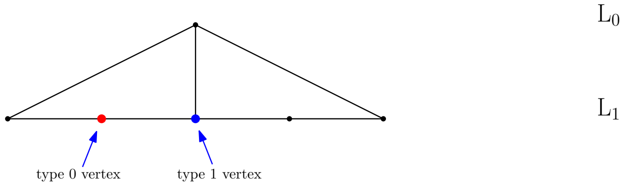

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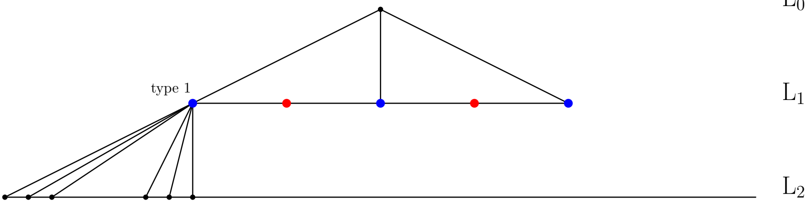

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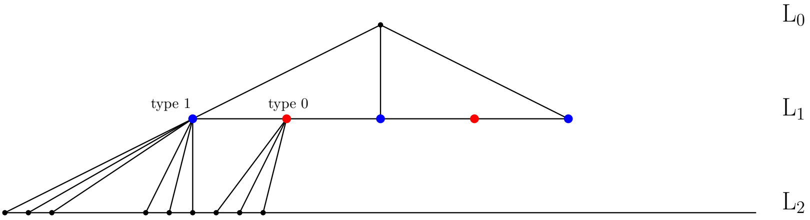

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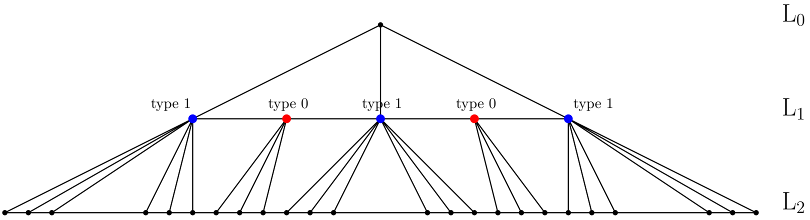

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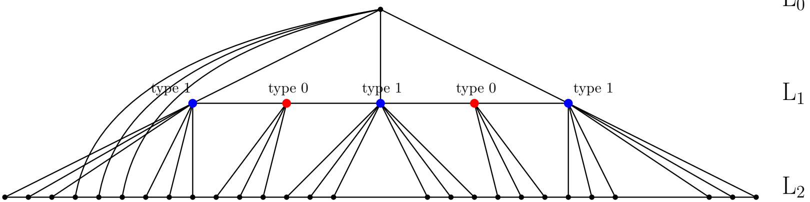

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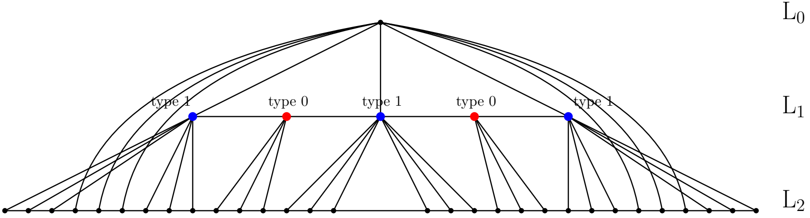

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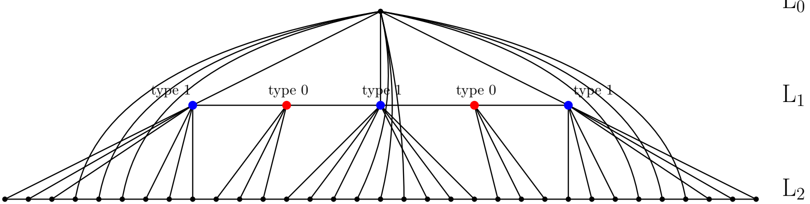

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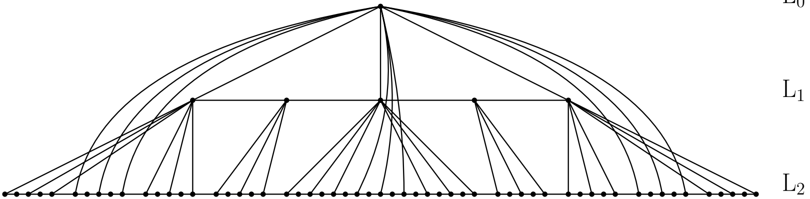

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EHF layered wheel

Remark

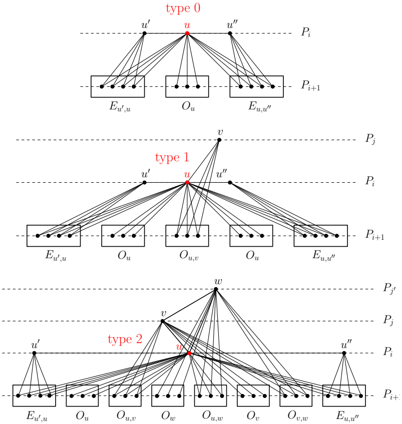
EHF layered wheel **contains triangle**.

EHF layered wheel

Remark

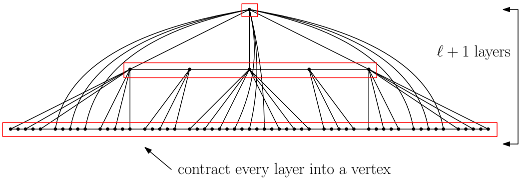
EHF layered wheel **contains triangle**.

- The first two layers are similar as for TTF layered wheel
- Three types of vertices in $G_{\ell,k}$:



Properties of layered wheels

- the shortest hole can be of arbitrarily length (at least 4)
- the **tree-width** can be arbitrarily large



- even the **rank-width** can be arbitrarily large

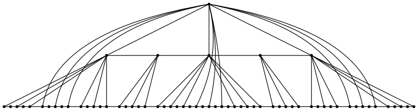
PART 2

Bounding the tree-width

Bounding the tree-width

Remark

To reach tree-width ℓ, layered wheel needs much more than 2^{ℓ} vertices.



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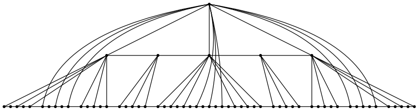

- Could it be that the tree-width is "small" in some sense?

Bounding the tree-width

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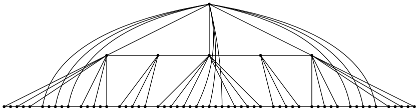

- Could it be that the tree-width is "small" in some sense?

Lemma (S., Trotignon, 2019+)

The tree-width of layered wheel on n vertices is in $O(\log n)$.

A conjecture

Conjecture

There exists a constant c such that the tree-width of any n-vertex (theta, triangle)-free graph is in $O(\log n)$.

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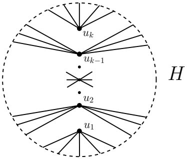
- If the conjecture is true, then many graph problems are poly-time solvable (it is indeed in $O(2^{tw})$).

An attempt toward the conjectures

Partial result



span wheel of order k

SPAN-WHEEL-NUMBER $\zeta(G)$:

the order of the largest span wheel in G

An attempt toward the conjectures

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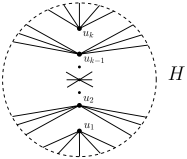

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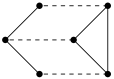
Theorem (S., Trotignon, 2019+)

- *The tree-width of any (theta, triangle)-free graph is in* $O\left(\zeta(G)^{o(1)}\right)$
- *idem for (even hole, K_4, pyramid)-free graph*


pyramid

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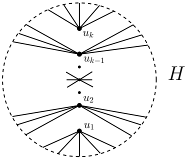

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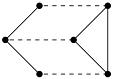
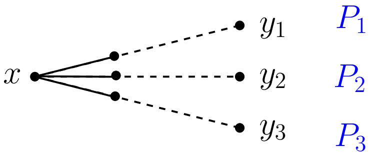


pyramid

but... $\zeta(G)$ can be up to $\frac{n}{2} - 1$ ☹

A consequence

An m-Spider, $m \geq 1$ is a graph consists of three internally-vertex-disjoint chordless paths P_1, P_2, P_3, each of length m



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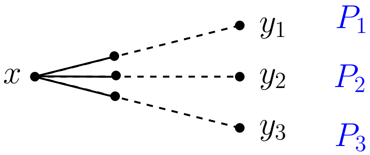

Theorem (S., Trotignon, 2019+)

Let $m \geq 1$. There exists a constant c such that any (theta, triangle, m-spider)-free graph G has tree-width $O\left(m^{o(1)}\right)$.

- Any *span-wheel* in G of $\geq \lfloor \frac{3m}{2} \rfloor$ centers contains an m-spider.

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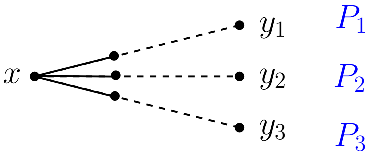

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- The theorem is best possible in some sense
- It was conjectured that: α is poly-time computable for **spider-free** graphs, also for **theta-free** graphs

Open Problem

A conjecture

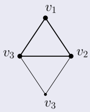
Conjecture

- *There exists a constant c such that the tree-width of any n-vertex (theta, triangle)-free graph is in* $O(\log n)$*.*
- *idem for (even hole,* K_4*)-free graphs*

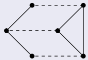
Another conjecture

Remark

EHF layered wheel contains **none** *of the following:*



diamond



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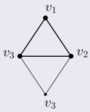

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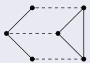

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- *There exists a constant c such that the tree-width of any (even hole, K_4, diamond)-free graphs is at most c.*

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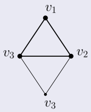

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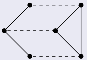

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- *There exists a constant c such that the tree-width of any (even hole, K_4, diamond)-free* graphs *is at most c.*
- *idem for (even hole, K_4, pyramid)-free* graphs

One more conjecture

Remark

Layered wheel contains **none** *of the following:*

- *large clique*
- *large bi-clique* ($K_{s,t}$) *(even as a non-induced subgraph)*
- *large grid or line graph of a grid, or its subdivision*
- *large wall or line graph of a wall, or its subdivision*

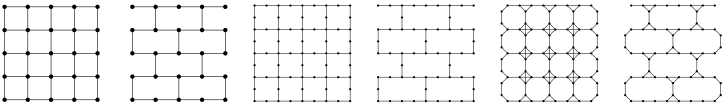

Figure: A grid, a wall, a subdivision of the former and its line graphs

One more conjecture

The grid-minor theorem:

- If G has huge tree-width then G must contain a **large grid** as a minor.

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Conjecture

If G has huge tree-width, then G must contain as an induced subgraph:

- *a big clique*
- *a big complete bipartite graph*
- *a big grid, possibly subdivided*
- *a big wall, possibly subdivided*
- *a big line graph of a subdivided wall*
- *layered wheels or variation of them*

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— The End —

References

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